Exponential Stochastic Cellular Automata for Massively Parallel Inference

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Abstract

We propose an embarrassingly parallel, memory efficient inference algorithm for latent variable models in which the complete data likelihood is in the exponential family. The algorithm is a stochastic cellular automaton and converges to a valid *maximum a posteriori* fixed point. Applied to latent Dirichlet allocation we find that our algorithm is over an order of magnitude faster than the fastest current approaches. A simple C++/MPI implementation on a 4-node cluster samples 570 million tokens per second. We process 3 billion documents and achieve predictive power competitive with collapsed Gibbs sampling and variational inference.

1 Introduction

In the past decade, frameworks such as stochastic gradient descent (SGD) [18] and map-reduce [5] have enabled machine learning algorithms to scale to larger and larger datasets. However, these frameworks are not always applicable to Bayesian latent variable models with rich statistical dependencies and intractable gradients. Variational methods [12] and Markov chain Monte-Carlo (MCMC) [7] have thus become the *sine qua non* for inferring the posterior in these models.

Sometimes—due to the concentration of measure phenomenon associated with large sample sizes—computing the full posterior is unnecessary and *maximum a posteriori* (MAP) estimates suffice. It is hence tempting to employ gradient descent, but for latent variable models such as latent Dirichlet allocation (LDA), calculating gradients involves expensive expectations over rich sets of variables [17]. MCMC is an appealing alternative, but algorithms such as the Gibbs sampler are inherently sequential and the extent to which they can be parallelized depends heavily upon how the structure of the statistical model interacts with the data. For instance, chromatic sampling [8] is infeasible for LDA, due to its dependence structure. Instead, we employ stochastic cellular automata (SCA), which like conventional cellular automata are massively parallel, but with stochastic updates.

We propose exponential SCA (ESCA) for inference in latent variable models with complete data likelihood in the exponential family. ESCA is embarassingly parallel because it is an SCA, and has a minimal memory footprint because it stores only the data and the sufficient statistics (by the very definition of sufficient statistics, the footprint cannot be further reduced). In contrast, variational approaches such as stochastic variational inference (SVI) [11] require storing the variational parameters, while MCMC-based methods, such as YahooLDA [20] require storing the latent variable assignments. Furthermore, our algorithm employs double-buffering for lock-free parameter updates (assuming atomic increments) while enabling the use of approximate counters. Thus, we substantially reduce memory costs and communication requirements in distributed enviroments.

2 Exponential SCA

Stochastic cellular automata (SCA), also known as probabilistic cellular automata, or locally-interacting Markov chains, are a stochastic version of a discrete-time, discrete-space dynamical system in which a noisy local update rule is homogeneously and synchronously applied to every site of a discrete space. They have been studied in statistical physics, mathematics, and computer science, and some progress has been made toward understanding their ergodicity and equilibrium properties. A recent survey [14] is an excellent introduction to the subject, and a dissertation [13] contains a comprehensive and precise presentation of SCA. Formally, the automaton, is given by an evolution function $\Phi : \mathcal{S} \longrightarrow \mathcal{S}$ over the state space $\mathcal{S} = \mathcal{Z} \longrightarrow C$ which is a mapping from the space of cell identifiers Z to cell values C. The global evolution function applies a local function $\phi_{\mathfrak{z}}(c_1, c_2, \cdots, c_r) \mapsto c$ s.t. $c_i = s(\mathfrak{z}_i)$ to every cell $\mathfrak{z} \in \mathcal{Z}$. That is, ϕ examines the values of each of the neighbors of cell $\mathfrak{z}$ and then stochastically computes a new value c. The dynamics begin with a state $s_0 \in S$ that can be configured using the data or a heuristic. Exponential SCA (ESCA) is based on SCA but achieves better computational efficiency by exploiting the structure of the sufficient statistics for latent variable models in which the complete data likelihood is in the exponential family. Most importantly, the local update function ϕ for each cell depends only upon the sufficient statistics and thus does *not* scale linearly with the number of neighbors.

2.1 Latent Variable Exponential Family

Latent variable models are useful when reasoning about partially observed data such as collections of text or images in which each *i.i.d.* data point is a document or image. Since the same local model is applied to each data point, they have the following form

$$p(\mathbf{z}, \mathbf{x}, \eta) = p(\eta) \prod_i p\left(z_i, x_i | \eta\right). \tag{1}$$

Our goal is to obtain a MAP estimate for the parameters η that explain the data $\mathbf{x}$ through the latent variables $\mathbf{z}$. To expose maximum parallelism, we want each cell in the automaton to correspond to a data point and its latent variable. However, this is problematic because in general all latent variables depend on each other via the global parameters η and a naive approach to updating a single cell would then require examining every other cell in the automaton.

Fortunately, if we further suppose that the complete data likelihood is in the exponential family, i.e., $p(z_i, x_i|\eta) = \exp\left(\langle T\left(z_i, x_i\right), \eta\rangle - g(\eta)\right)$ then the sufficient statistics are given by $T(\mathbf{z}, \mathbf{x}) = \sum_i T(z_i, x_i)$ and we can thus express any estimator of interest as a function of just $T(\mathbf{z}, \mathbf{x})$ which factorizes over the data. Further, when employing expectation maximization (EM), the M-step is possible in closed form for many members of the exponential family. This allows us to reformulate the cell level updates to depend only upon the sufficient statistics instead of the neighboring cells. The idea is that, unlike SCA (or MCMC in general) which produces a sequence of states that correspond to complete variable assignments $s^0, s^1, \ldots$ via a transition kernel $q(s^{t+1}|s^t)$, ESCA produces a sequence of sufficient statistics $T^0, T^1, \ldots$ directly via an evolution function $\Phi(T^t) \mapsto T^{t+1}$.

2.2 Stochastic EM

Before we present ESCA, we first describe stochastic EM (SEM). Suppose we want the MAP estimate for η, $\max_\eta p(\mathbf{x}, \eta) = \max_\eta \int p(\mathbf{z}, \mathbf{x}, \eta)\mu(d\mathbf{z})$ and employ expectation maximization (EM):

E-step Compute in parallel $p(z_i|x_i, \eta^{(t)})$.
M-step Find $\eta^{(t+1)}$ that maximizes the expected log-likelihood with respect to the conditional

$$\eta^{(t+1)} = \arg\max_\eta \mathbb{E}_{\mathbf{z}|\mathbf{x},\eta^{(t)}}\left[\log p(\mathbf{z}, \mathbf{x}, \eta)\right] = \xi^{-1}\left(\frac{1}{n + n_0}\sum_i \mathbb{E}_{\mathbf{z}|\mathbf{x},\eta^{(t)}}\left[T(z_i, x_i)\right] + T_0\right)$$

where $\xi(\eta) = \nabla g(\eta)$ is invertible as $\nabla^2 g(\theta) \succ 0$ and n_0, T_0 parametrize the conjugate prior. Although EM exposes substantial parallelism, it is difficult to scale, since the dense structure $p(z_i|x_i, \eta^{(t)})$ defines values for all possible outcomes for z and thus puts tremendous pressure on memory bandwidth. To overcome this we introduce sparsity by employing stochastic EM (SEM) [3]. SEM introduces an S-step after the E-step that replaces the full distribution with a single sample:

(a) Phase 1 (b) Phase 2

Figure 1: Efficient (re)use of buffers

S-step Sample $z_i^{(t)} \sim p(z_i|x_i;\eta^{(t)})$ in parallel.

Subsequently, we perform the M-step using the imputed data instead of the expectation. This simple modification overcomes the computational drawbacks of EM for cases in which sampling from $p(z_i|x_i;\eta^{(t)})$ is feasible. We can now employ fast samplers, such as the alias method, exploit sparsity, reduce CPU-RAM bandwidth while still maintaining massive parallelism. More importantly, the S-step also enables all three steps to now be expressed in terms of the current sufficient statistics. This enables distributed and parallel implementations that efficiently execute on an SCA.

2.3 ESCA for Latent Variable Models

We now present ESCA as SEM on an SCA in which each cell corresponds to a data point with its associated latent variables. Define an SCA over the state space $\mathcal{S}$ of the form $\mathcal{S} = \mathcal{Z} \longrightarrow \mathcal{K} \times \mathcal{X}$, where $\mathcal{Z}$ is the set of cell identifiers (e.g., one per data point), $\mathcal{K}$ is the domain of latent variables, and $\mathcal{X}$ is the domain of the observed data. The initial state s_0 is the map defined as follows: for every data point, we associate a cell z to the pair (k_z, x) where k_z is chosen at random from $\mathcal{K}$ and independently from $k_{z'}$ for all $z' \neq z$. This gives us the initial state $s_0 = z \mapsto (k_z, x)$.

We now need to describe the evolution function Φ. For state s and cell z define the distribution:

$$p_z(k|s) = f(z, T(s)) \tag{2}$$

Assuming that $s(z) = (k, x)$ and that k' is a sample from p_z (hence the name "stochastic" cellular automaton) we define the local update function as $\phi(s, z) = (k', x)$ where $s(z) = (k, x)$ and $k' \sim p_z(\,\cdot\,|s)$. That is, the observed data remain unchanged, but we choose a new latent variable according to the distribution p_z induced by the state. We obtain the evolution function of the stochastic cellular automaton by applying the function ϕ uniformly on every cell $\Phi(s) = z \mapsto \phi(s, z)$. Finally, the SCA algorithm simulates the evolution function Φ starting with s_0. We remark that ESCA converges weakly to a distribution with mean equal to some root of the score function $(\nabla_\eta \log p(x_i;\eta))$ and thus a MAP fixed point. See Appendix D for details.

Our implementation has two copies of the data structure containing sufficient statistics $T^{(0)}$ and $T^{(1)}$. We do not compute the values $T(\mathbf{z}, \mathbf{x})$ but maintain their sum as we impute values of the cells/latent variables. During iteration $2t$ of the evolution function, we apply Φ by reading from $T^{(0)}$ and incrementing $T^{(1)}$ as we sample the latent variables (Figure 1). Then in the next iteration $2t+1$ we reverse the roles of data structure, i.e. read from $T^{(1)}$ and increment $T^{(0)}$. See Algorithm 1.

Algorithm 1 ESCA

```
Randomly initialize each cell
for t = 0 → num iterations do
    for all cell z independently in parallel do
        Read sufficient statistics from T^(t mod 2)
        Compute stochastic updates using p_z(k|s)
        Write sufficient statistics to T^((t+1) mod 2)
    end for
end for
```

Use of such read/write buffers offer a virtually lock-free (assuming atomic increments) implementation scheme for ESCA and is analogous to double-buffering in computer graphics. Although there is a synchronization barrier after each round, its effect is mitigated because each cell's work depends only upon the sufficient statistics and thus does the same amount of work. Therefore, evenly balancing the work load across computation nodes is trivial, even for a heterogeneous cluster.

3 ESCA for LDA

Latent Dirichlet allocation (LDA) [1] is a must-have for analytic platforms and consequently needs to scale. LDA models each document m of M documents as a distribution θ_m over K topics. A topic k is a distribution ϕ_k over V vocabulary words. A document m comprises N_m words w_{mn} each with a latent variable z_{mn} indicating a topic assignment. Both distributions θ_m and ϕ_k have a Dirichlet prior, parameterized respectively with a constant α and β. See Appendix B for details.

ESCA simulates the inference steps of SEM which we derive for LDA in Appendix B. For LDA, the state space is $\mathcal{S} = \mathcal{Z} \longrightarrow \mathcal{K} \times \mathcal{M} \times \mathcal{V}$ where $\mathcal{Z}$ is the set of cell identifiers (one per token in our corpus), $\mathcal{K}$ is a set of K topics, $\mathcal{M}$ is a set of M document identifiers, and $\mathcal{V}$ is a set of V identifiers for the vocabulary words. From this we obtain the full conditional of LDA for line 5 of Algorithm 1

$$p_z(k|s) \propto \boxed{(D_{mk} + \alpha) \times \frac{W_{kv} + \beta}{T_k + \beta V}} \tag{3}$$

where $D_{mk} = \Big|\{\, z_{mn} \mid z_{mn} = k \,\}\Big|$, $W_{kv} = \Big|\{\, z_{mn} \mid w_{mn} = v,\ z_{mn} = k \,\}\Big|$, and $T_k = \sum_{v=1}^{V} W_{kv}$.

It reassuring to see that the boxed region of Equation 3 is similar to respective formulas in collapsed Gibbs sampling (CGS) [9] and collapsed variational Bayes (CVB0) [21]. For LDA, ESCA implicitly performs SGD with Frank-Wolfe updates, alluding to a convergence rate (Appendix C).

4 Experiments

Software & hardware All algorithms are implemented in C++11. We implement multithreaded parallelization within a node using the work-stealing Fork/Join framework, and the distribution across multiple nodes using the process binding to a socket over MPI. We implement ESCA with a sparse arrays D counting topics per documents and Vose's alias method to draw from discrete distributions. We run our experiments on a small cluster of 4 nodes connected through 10Gb/s Ethernet. Each node has two 9-core Intel Xeon E5 processors for a total of 36 hardware threads per node.

Datasets We employ three datasets: PubMed abstracts (141,043 vocabulary words, 8.2 million documents, 737 million tokens), Wikipedia (210,223 words, 6.6 million documents, 1.1 billion tokens) and a large proprietary dataset (140,000 words, 3 billion documents, and 171 billion tokens).

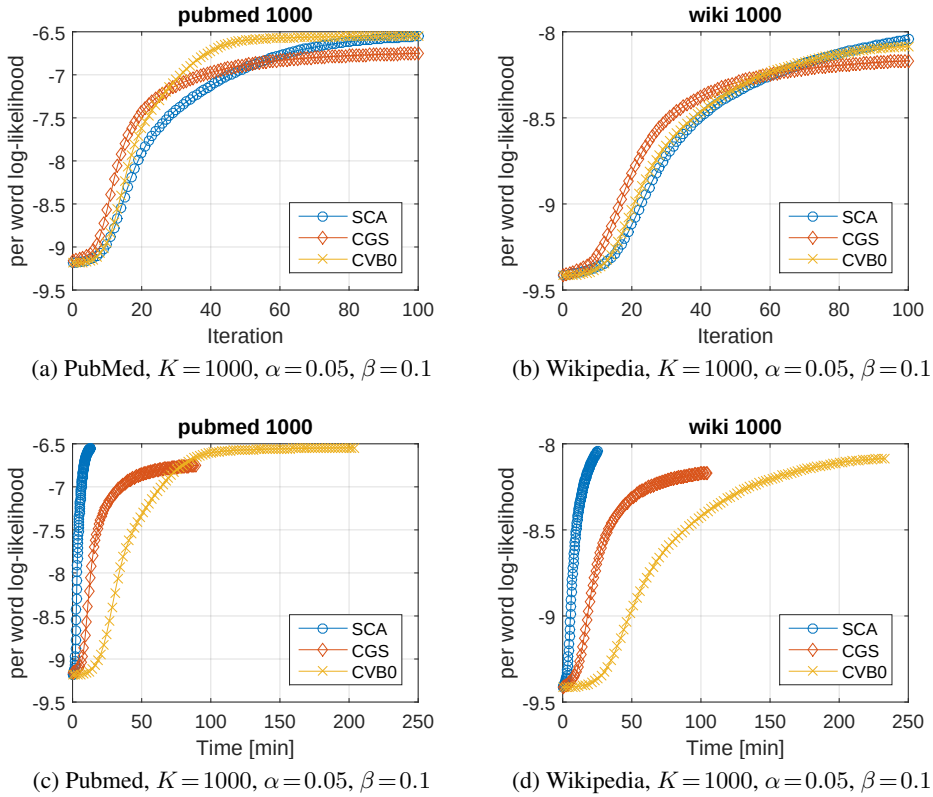
Evaluation To evaluate the proposed method we use predicting power as a metric by calculating the per-word log-likelihood (equivalent to negative log of perplexity) on 10,000 held-out documents conditioned on the trained model. We set $K = 1000$ to demonstrate performance for a large number of topics. The hyper parameters are set as $\alpha = 50/K$ and $\beta = 0.1$ as suggested in [10]; other systems such as YahooLDA and Mallet also use this as the default parameter setting. The results are in Figure 2 and additional experiments are in Appendix H. Finally, for the large dataset, our implementation of ESCA (only 300 lines of C++) processes 570 million tokens per second (tps) on our modest 4-node cluster. In comparison, some of the best existing systems achieve 112 million tps (F+LDA, personal communication) and 60 million tps (lightLDA) [25]. See Table 1 for details.

Table 1: Comparison with existing scalable LDA frameworks.

Method	Dataset	Infrastructure	Processing speed
YahooLDA [20]	140K vocab, 8.2M docs, 797M tokens	10 machines 2010	12.87M tokens/s
lightLDA [25]	50K vocab, 1.2B docs, 200B tokens	24 machines 2014	60M tokens/s
F+LDA [24]	1M vocab, 29M docs, 1.5B tokens	32 machines 2014	110M tokens/s
ESCA	**210K vocab, 6B docs, 128B tokens**	**8 Amazon c4.8x large**	**503M tokens/s**

Discussion

We proposed an embarassingly parallel, memory efficient MAP inference algorithm that executes on an SCA and applies to a large class of latent variable models. Our algorithm exposes many system level optimizations such as approximate counters, and outperforms current best approaches.



(a) PubMed, $K = 1000$, $\alpha = 0.05$, $\beta = 0.1$

(b) Wikipedia, $K = 1000$, $\alpha = 0.05$, $\beta = 0.1$

(c) Pubmed, $K = 1000$, $\alpha = 0.05$, $\beta = 0.1$

(d) Wikipedia, $K = 1000$, $\alpha = 0.05$, $\beta = 0.1$

Figure 2: Evolution of log likelihood on Wikipedia and Pubmed over number of iterations and time.

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A (Stochastic) EM in General

Expectation-Maximization (EM) is an iterative method for finding the maximum likelihood or *maximum a posteriori* (MAP) estimates of the parameters in statistical models when data is only partially, or when model depends on unobserved latent variables. This section is inspired from `http://www.ece.iastate.edu/~namrata/EE527_Spring08/emlecture.pdf`

We derive EM algorithm for a very general class of model. Let us define all the quantities of interest.

Table 2: Notation

Symbol	Meaning
$\mathbf{x}$	Observed data
$\mathbf{z}$	Unobserved data
$(\mathbf{x}, \mathbf{z})$	Complete data
$f_{\mathbf{X};\eta}(\mathbf{x};\eta)$	marginal observed data density
$f_{\mathbf{Z};\eta}(\mathbf{z};\eta)$	marginal unobserved data density
$f_{\mathbf{X},\mathbf{Z};\eta}(\mathbf{x},\mathbf{z};\eta)$	complete data density/likelihood
$f_{\mathbf{Z}\|\mathbf{X};\eta}(\mathbf{z}\|\mathbf{x};\eta)$	conditional unobserved-data (missing-data) density.

Objective: To maximize the marginal log-likelihood or posterior, i.e.

$$L(\eta) = \log f_{\mathbf{X};\eta}(\mathbf{x};\eta). \tag{4}$$

Assumptions:

1. z_i are independent given η. So

$$f_{\mathbf{Z};\eta}(\mathbf{z};\eta) = \prod_{i=1}^{N} f_{Z_i;\eta}(z_i;\eta), \tag{5}$$

2. x_i are independent given missing data z_i and η. So

$$f_{\mathbf{X},\mathbf{Z};\eta}(\mathbf{x},\mathbf{z};\eta) = \prod_{i=1}^{N} f_{X_i,Z_i;\eta}(x_i,z_i;\eta). \tag{6}$$

As a consequence we obtain:

$$f_{\mathbf{Z}|\mathbf{X};\eta}(\mathbf{z}|\mathbf{x};\eta) = \prod_{i=1}^{N} f_{Z_i|X_i;\eta}(z_i|x_i;\eta), \tag{7}$$

Now,

$$L(\eta) = \log f_{\mathbf{X};\eta}(\mathbf{x};\eta) = \log f_{\mathbf{X},\mathbf{Z};\eta}(\mathbf{x},\mathbf{z};\eta) - \log f_{\mathbf{Z}|\mathbf{X};\eta}(\mathbf{z}|\mathbf{x};\eta) \tag{8}$$

or, summing across observations,

$$L(\eta) = \sum_{i=1}^{N} \log f_{X_i;\eta}(x_i;\eta) = \sum_{i=1}^{N} \log f_{X_i,Z_i;\eta}(x_i,z_i;\eta) - \sum_{i=1}^{N} \log f_{Z_i|X_i;\eta}(z_i|x_i;\eta). \tag{9}$$

Let us take the expectation of the above expression with respect to $f_{Z_i|X_i;\eta}(z_i|x_i;\eta_p)$, where we choose $\eta = \eta_p$:

$$\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta}\left[\log f_{X_i;\eta}(x_i;\eta)|x_i;\eta_p\right] \\
&\quad = \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta}\left[\log f_{X_i,Z_i;\eta}(x_i,z_i;\eta)|x_i;\eta_p\right] - \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta}\left[\log f_{Z_i|X_i;\eta}(z_i|x_i;\eta)|x_i;\eta_p\right]
\end{aligned} \tag{10}$$

Since $L(\eta) = \log f_{\mathbf{X};\eta}(\mathbf{x};\eta)$ does not depend on $\mathbf{z}$, it is invariant for this expectation. So we recover:

$$\begin{aligned} L(\eta) &= \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta} \left[\log f_{X_i,Z_i;\eta}(x_i, z_i; \eta)|x_i; \eta_p\right] - \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta} \left[\log f_{Z_i|X_i;\eta}(z_i|x_i; \eta)|x_i; \eta_p\right] \\ &= Q(\eta|\eta_p) - H(\eta|\eta_p). \end{aligned} \tag{11}$$

Now, (11) may be written as

$$Q(\eta|\eta_p) = L(\eta) + \underbrace{H(\eta|\eta_p)}_{\leq H(\eta_p|\eta_p)} \tag{12}$$

Here, observe that $H(\eta|\eta_p)$ is maximized (with respect to η) by $\eta = \eta_p$, i.e.

$$H(\eta|\eta_p) \leq H(\eta_p|\eta_p) \tag{13}$$

Simple proof using Jensen's inequality.

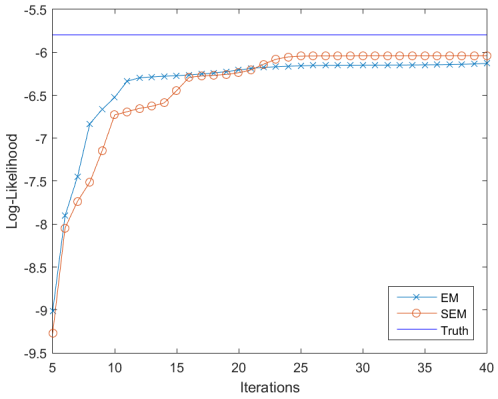
As our objective is to maximize $L(\eta)$ with respect to η, if we maximize $Q(\eta|\eta_p)$ with respect to η, it will force $L(\eta)$ to increase. This is what is done repetitively in EM. To summarize, we have:

E-step : Compute $f_{Z_i|X_i;\eta}(z_i|x_i; \eta_p)$ using current estimate of $\eta = \eta_p$.

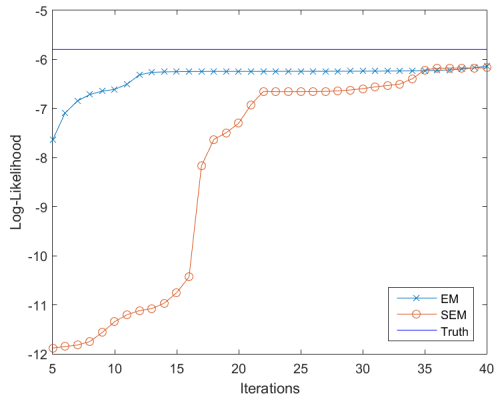
M-step : Maximize $Q(\eta|\eta_p)$ to obtain next estimate η_{p+1}.

Now assume that the complete data likelihood belongs to the exponential family, i.e.

$$f_{X_i,Z_i;\eta}(x_i, z_i; \eta) = \exp\left(\langle T\left(z_i, x_i\right), \eta\rangle - g(\eta)\right) \tag{14}$$



(a) Same initialization



(b) Bad initialization for SEM

Figure 3: Performance of SEM

then

$$
\begin{aligned}
Q(\eta|\eta_p) &= \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta} \left[\log f_{X_i,Z_i;\eta}(x_i, z_i; \eta)|x_i; \eta_p\right] \\
&= \sum_{i=1}^{N} \mathbb{E}_{Z_i|X_i;\eta} \left[\langle T\left(z_i, x_i\right), \eta\rangle - g(\eta)|x_i; \eta_p\right]
\end{aligned} \tag{15}
$$

To find the maximizer, differentiate and set it to zero:

$$
\frac{1}{N}\sum_i \mathbb{E}_{Z_i|X_i;\eta} \left[\langle T\left(z_i, x_i\right), \eta\rangle \, |x_i; \eta_p\right] = \frac{dg(\eta)}{d\eta} \tag{16}
$$

and one can obtain the maximizer by solving this equation.

Stochastic EM (SEM) introduces an additional simulation after the E-step that replaces the full distribution with a single sample:

S-step Sample $z_i \sim f_{Z_i|X_i;\eta}(z_i|x_i; \eta_p)$

This essentially means we replace $\mathbb{E}[\cdot]$ with an empirical estimate. Thus, instead of solving (16), we simply have:

$$
\frac{1}{N}\sum_i T(z_i, x_i) = \frac{dg(\eta)}{d\eta}. \tag{17}
$$

Computing and solving this system of equations is considerably easier than (16).

Now to demonstrate that SEM is well behaved and works in practice, we run a small experiment. Consider the problem of estimating the parameters of a Gaussian mixture. We choose a 2-dimensional Gaussian with $K = 30$ clusters and 100,000 training points and 1,000 test points. We run EM and SEM with the following initialization:

- Both SEM and EM are provided the same initialization.
- SEM is deliberately provided a bad initialization, while EM is not.

The log-likelihood on the heldout test set is shown in Figure 3.

B (S)EM Derivation for LDA

We derive an EM procedure for LDA.

B.1 LDA Model

In LDA, we model each document m of a corpus of M documents as a distribution θ_m that represents a mixture of topics. There are K such topics, and we model each topic k as a distribution ϕ_k over the vocabulary of words that appear in our corpus. Each document m contains N_m words w_{mn} from a vocabulary of size V, and we associate a latent variable z_{mn} to each of the words. The latent variables can take one of K values that indicate which topic the word belongs to. We give each of the distributions θ_m and ϕ_k a Dirichlet prior, parameterized respectively with a constant α and β. More concisely, LDA has the following mixed density.

$$p(\boldsymbol{w}, \boldsymbol{z}, \boldsymbol{\theta}, \boldsymbol{\phi}) = \left[\prod_{m=1}^{M}\prod_{n=1}^{N_m} \mathrm{Cat}(w_{mn} \mid \phi_{z_{mn}})\ \mathrm{Cat}(z_{mn} \mid \theta_m)\right]\left[\prod_{m=1}^{M} \mathrm{Dir}(\theta_m \mid \alpha)\right]\left[\prod_{k=1}^{K} \mathrm{Dir}(\phi_k \mid \beta)\right] \tag{18}$$

The choice of a Dirichlet prior is not a coincidence: we can integrate all of the variables θ_m and ϕ_k and obtain the following closed form solution.

$$p(\boldsymbol{w}, \boldsymbol{z}) = \left[\prod_{m=1}^{M} \mathrm{Pol}\big(\{z_{m'n} \mid m' = m\}, K, \alpha\big)\right]\left[\prod_{k=1}^{K} \mathrm{Pol}\big(\{w_{mn} \mid z_{mn} = k\}, V, \beta\big)\right] \tag{19}$$

where Pol is the Polya distribution

$$\mathrm{Pol}(S, X, \eta) = \frac{\Gamma(\eta\, K)}{\Gamma(|S| + \eta\, X)} \prod_{x=1}^{X} \frac{\Gamma\big(\big|\{z \mid z \in S, z = x\}\big| + \eta\big)}{\Gamma(\eta)} \tag{20}$$

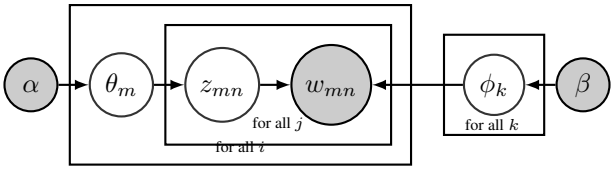


Figure 4: LDA Graphical Model

Algorithm 2 LDA Generative Model

input: $\boldsymbol{\alpha}, \boldsymbol{\beta}$

for $k = 1 \to K$ **do**
  Choose topic $\boldsymbol{\phi}_k \sim \mathsf{Dir}(\boldsymbol{\beta})$
end for
for all document m in corpus D **do**
  Choose a topic distribution $\boldsymbol{\theta}_m \sim \mathsf{Dir}(\boldsymbol{\alpha})$
  for all word index n from 1 to N_m **do**
    Choose a topic $z_{mn} \sim \mathsf{Categorical}(\boldsymbol{\theta}_m)$
    Choose word $w_{mn} \sim \mathsf{Categorical}(\boldsymbol{\phi}_{z_{mn}})$
  end for
end for

The joint probability density can be expressed as:

$$\begin{aligned} p(W, Z, \theta, \phi|\alpha, \beta) &= \left[\prod_{k=1}^{K} p(\phi_k|\beta)\right]\left[\prod_{m=1}^{M} p(\theta_m|\alpha) \prod_{n=1}^{N_m} p(z_{mn}|\theta_m) p(w_{mn}|\phi_{z_{mn}})\right] \\ &\propto \left[\prod_{k=1}^{K}\prod_{v=1}^{V} \phi_{kv}^{\beta-1}\right]\left[\prod_{m=1}^{M}\left(\prod_{k=1}^{K} \theta_{mk}^{\alpha-1}\right)\prod_{n=1}^{N_m} \theta_{m z_{mn}} \phi_{z_{mn} w_{mn}}\right] \end{aligned} \tag{21}$$

B.2 Expectation Maximization

We begin by marginalizing the latent variable Z and finding the lower bound for the likelihood/posterior:

$$\begin{aligned}
\log p(W, \theta, \phi|\alpha, \beta) &= \log \sum_Z p(W, Z, \theta, \phi|\alpha, \beta) \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N_m} \log \sum_{k=1}^{K} p(z_{mn} = k|\theta_m) p(w_{mn}|\phi_k) \\
&\qquad + \sum_{k=1}^{K} \log p(\phi_k|\beta) + \sum_{m=1}^{M} \log p(\theta_m|\alpha) \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N_m} \log \sum_{k=1}^{K} q(z_{mn} = k|w_{mn}) \frac{p(z_{mn} = k|\theta_m) p(w_{mn}|\phi_k)}{q(z_{mn} = k|w_{mn})} \\
&\qquad + \sum_{k=1}^{K} \log p(\phi_k|\beta) + \sum_{m=1}^{M} \log p(\theta_m|\alpha) \\
\text{(Jensen Inequality)} \qquad &\geq \sum_{m=1}^{M} \sum_{n=1}^{N_m} \sum_{k=1}^{K} q(z_{mn} = k|w_{mn}) \log \frac{p(z_{mn} = k|\theta_m) p(w_{mn}|\phi_k)}{q(z_{mn} = k|w_{mn})} \\
&\qquad + \sum_{k=1}^{K} \log p(\phi_k|\beta) + \sum_{m=1}^{M} \log p(\theta_m|\alpha)
\end{aligned} \tag{22}$$

Let us define the following functional:

$$\begin{aligned}
F(q, \theta, \phi) := &- \sum_{m=1}^{M} \sum_{n=1}^{N_m} D_{KL}(q(z_{mn}|w_{mn})||p(z_{mn}|w_{mn}, \theta_m, \phi)) \\
&+ \sum_{m=1}^{M} \sum_{n=1}^{N_m} p(w_{mn}|\theta_m, \phi) + \sum_{k=1}^{K} \log p(\phi_k|\beta) + \sum_{m=1}^{M} \log p(\theta_m|\alpha)
\end{aligned} \tag{23}$$

B.2.1 E-Step

In the E-step, we fix θ, ϕ and maximize F for q. As q appears only in the KL-divergence term, it is equivalent to minimizing the KL-divergence between $q(z_{mn}|w_{mn})$ and $p(z_{mn}|w_{mn}, \theta_m, \phi)$. We know that for any distributions f and g the KL-divergence is minimized when $f = g$ and is equal to 0. Thus, we have

$$\begin{aligned}
q(z_{mn} = k|w_{mn}) &= p(z_{mn} = k|w_{mn}, \theta_m, \phi) \\
&= \frac{\theta_{mk} \phi_{k w_{mn}}}{\sum_{k'=1}^{K} \theta_{mk'} \phi_{k' w_{mn}}}
\end{aligned} \tag{24}$$

For simplicity of notation, let us define

$$\boxed{q_{mnk} = \frac{\theta_{mk} \phi_{k w_{mn}}}{\sum_{k'=1}^{K} \theta_{mk'} \phi_{k' w_{mn}}}} \tag{25}$$

B.2.2 M-Step

In the E-step, we fix q and maximize F for θ, ϕ. As this will be a constrained optimization (θ and ϕ must lie on simplex), we use standard constrained optimization procedure of Lagrange multipliers.

The Lagrangian can be expressed as:

$$\begin{aligned}\mathcal{L}(\theta,\phi,\lambda,\mu) &= \sum_{m=1}^{M}\sum_{m=1}^{N_m}\sum_{k=1}^{K} q(z_{mn}=k|w_{mn})\log\frac{p(z_{mn}=k|\theta_m)p(w_{mn}|\phi_k)}{q(z_{mn}=k|w_{mn})} + \sum_{k=1}^{K}\log p(\phi_k|\beta) \\ &\quad + \sum_{m=1}^{M}\log p(\theta_m|\alpha) + \sum_{k=1}^{K}\lambda_k\left(1-\sum_{v=1}^{V}\phi_{kv}\right) + \sum_{m=1}^{M}\mu_i\left(1-\sum_{k=1}^{K}\theta_{mk}\right) \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N_m}\sum_{k=1}^{K} q_{mnk}\log\theta_{mk}\phi_{kw_{mn}} + \sum_{k=1}^{K}\sum_{v=1}^{V}(\beta_v-1)\log\phi_{kv} + \sum_{m=1}^{M}\sum_{k=1}^{K}(\alpha_k-1)\log\theta_{mk} \\ &\quad + \sum_{k=1}^{K}\lambda_k\left(1-\sum_{v=1}^{V}\phi_{kv}\right) + \sum_{m=1}^{M}\mu_m\left(1-\sum_{k=1}^{K}\theta_{mk}\right) + \text{const.}\end{aligned} \tag{26}$$

Maximizing θ Taking derivative with respect to θ_{mk} and setting it to 0, we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\theta_{mk}} = 0 &= \sum_{j=1}^{N_m}\frac{q_{mnk}+\alpha_k-1}{\theta_{mk}} - \mu_m \\ \mu_m\theta_{mk} &= \sum_{j=1}^{N_i} q_{mnk} + \alpha_k - 1\end{aligned} \tag{27}$$

After solving for μ_m, we finally obtain

$$\theta_{mk} = \frac{\sum_{n=1}^{N_m} q_{mnk}+\alpha_k-1}{\sum_{k'=1}^{K}\sum_{j=1}^{N_m} q_{mnk'}+\alpha_{k'}-1} \tag{28}$$

Note that $\sum_{k'=1}^{K} q_{mnk'} = 1$, we reach at the optimizer:

$$\boxed{\theta_{mk} = \frac{1}{N_m + \sum(\alpha_{k'}-1)}\left(\sum_{n=1}^{N_m} q_{mnk} + \alpha_k - 1\right)} \tag{29}$$

Maximizing ϕ Taking derivative with respect to ϕ_{kv} and setting it to 0, we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_{kv}} = 0 &= \sum_{m=1}^{M}\sum_{n=1}^{N_m}\frac{q_{mnk}\delta(v-w_{mn})+\beta_v-1}{\phi_{kv}} - \lambda_k \\ \lambda_k\phi_{kv} &= \sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v-w_{mn}) + \beta_v - 1\end{aligned} \tag{30}$$

After solving for λ_k, we finally obtain

$$\phi_{kv} = \frac{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v-w_{mn})+\beta_v-1}{\sum_{v'=1}^{V}\sum_{m=1}^{M}\sum_{n=1}^{N_m}\delta(v'-w_{mn})+\beta_{v'}-1} \tag{31}$$

Note that $\sum_{v'=1}^{V}\delta(v'-w_{mn}) = 1$, we reach at the optimizer:

$$\boxed{\phi_{kv} = \frac{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v-w_{mn})+\beta_v-1}{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk} + \sum(\beta_{v'}-1)}} \tag{32}$$

B.3 Introducing Stochasticity

After performing the E-step, we add an extra simulation step, i.e. we draw and impute the values for the latent variables from its distribution conditioned on data and current estimate of the parameters. This means basically q_mnk gets transformed into $\delta(z_{mn}-\tilde{k})$ where $\tilde{k}$ is value drawn from the conditional distribution. Then we proceed to perform the M-step, which is even simpler now. To summarize SEM for LDA will have following steps:

E-step in parallel compute the conditional distribution locally:

$$q_{mnk} = \frac{\theta_{mk}\phi_{kw_{mn}}}{\sum_{k'=1}^{K}\theta_{mk'}\phi_{k'w_{ij}}} \tag{33}$$

S-step in parallel draw z_{mn} from the categorical distribution:

$$z_{mn} \sim \mathsf{Categorical}(q_{mn1}, ..., q_{mnK}) \tag{34}$$

M-step in parallel compute the new parameter estimates:

$$\begin{aligned} \theta_{mk} &= \frac{D_{mk} + \alpha_k - 1}{N_m + \sum(\alpha_{k'} - 1)} \\ \phi_{kv} &= \frac{W_{kv} + \beta_v - 1}{T_k + \sum(\beta_{v'} - 1)} \end{aligned} \tag{35}$$

where $D_{mk} = \Big|\big\{\, z_{mn} \mid z_{mn} = k \,\big\}\Big|$,
$W_{kv} = \Big|\big\{\, z_{mn} \mid w_{mn} = v,\, z_{mn} = k \,\big\}\Big|$, and
$T_k = \Big|\big\{\, z_{mn} \mid z_{mn} = k \,\big\}\Big| = \sum_{v=1}^{V} W_{kv}$.

C Equivalency between (S)EM and (S)GD for LDA

We study the equivalency between (S)EM and (S)GD for LDA. The deterministic case (EM and GD) has been studied for other models [23, 19]. We derive it for LDA here for completeness.

C.1 EM for LDA

EM for LDA can be summarized by follows:

E-Step

$$q_{mnk} = \frac{\theta_{mk}\phi_{kw_{mn}}}{\sum_{k'=1}^{K}\theta_{mk'}\phi_{k'w_{mn}}} \tag{36}$$

M-Step

$$\begin{aligned}\theta_{mk} &= \frac{1}{N_m + \sum(\alpha_{k'}-1)}\left(\sum_{n=1}^{N_m} q_{mnk} + \alpha_k - 1\right)\\ \phi_{kv} &= \frac{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v - w_{mn}) + \beta_v - 1}{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk} + \sum(\beta_{v'}-1)}\end{aligned} \tag{37}$$

C.2 GD for LDA

The joint probability density can be expressed as:

$$\begin{aligned}p(W, Z, \theta, \phi|\alpha, \beta) &= \left[\prod_{k=1}^{K} p(\phi_k|\beta)\right]\left[\prod_{m=1}^{M} p(\theta_m|\alpha)\prod_{n=1}^{N_m} p(z_{mn}|\theta_m)p(w_{mn}|\phi_{z_{mn}})\right]\\ &\propto \left[\prod_{k=1}^{K}\prod_{v=1}^{V}\phi_{kv}^{\beta-1}\right]\left[\prod_{m=1}^{M}\left(\prod_{k=1}^{K}\theta_{mk}^{\alpha-1}\right)\prod_{n=1}^{N_m}\theta_{mz_{mn}}\phi_{z_{mn}w_{mn}}\right]\end{aligned} \tag{38}$$

The log-probability of joint model with Z marginalized can be written as:

$$\begin{aligned}\log p(W, \theta, \phi|\alpha, \beta) &= \log\sum_{Z} p(W, Z, \theta, \phi|\alpha, \beta)\\ &= \sum_{m=1}^{M}\sum_{n=1}^{N_m}\log\sum_{k=1}^{K} p(z_{mn}=k|\theta_m)p(w_{mn}|\phi_k)\\ &\qquad + \sum_{k=1}^{K}\log p(\phi_k|\beta) + \sum_{m=1}^{M}\log p(\theta_m|\alpha)\\ &= \sum_{m=1}^{M}\sum_{n=1}^{N_m}\log\sum_{k=1}^{K}\theta_{mk}\phi_{kw_{mn}}\\ &\qquad + \sum_{m=1}^{M}\sum_{k=1}^{K}(\alpha_k-1)\log\theta_{mk} + \sum_{k=1}^{K}\sum_{v=1}^{V}(\beta_v-1)\log\phi_{kv}\end{aligned} \tag{39}$$

Gradient for topic per document Now take derivative with respect to θ_{mk}:

$$\begin{aligned}\frac{\partial\log p}{\partial\theta_{mk}} &= \sum_{j=1}^{N_m}\frac{\phi_{kw_{mn}}}{\sum_{k'=1}^{K}\theta_{mk'}\phi_{k'w_{mn}}} + \frac{\alpha_k-1}{\theta_{mk}}\\ &= \frac{1}{\theta_{mk}}\left(\sum_{n=1}^{N_m} q_{mnk} + \alpha_k - 1\right)\end{aligned} \tag{40}$$

Gradient for word per topic Now take derivative with respect to ϕ_{kv}:

$$\begin{aligned}\frac{\partial \log p}{\partial \phi_{kv}} &= \sum_{m=1}^{M}\sum_{n=1}^{N_m}\frac{\theta_{mk}\delta(v-w_{mn})}{\sum_{k'=1}^{K}\theta_{mk'}\phi_{k'w_{mn}}} + \frac{\beta_v - 1}{\phi_{kv}} \\ &= \frac{1}{\phi_{kv}}\left(\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v-w_{mn}) + \beta_v - 1\right)\end{aligned} \tag{41}$$

C.3 Equivalency

If we look at one step of EM:

For topic per document

$$\begin{aligned}\theta_{mk}^{+} &= \frac{1}{N_m + \sum(\alpha_{k'}-1)}\left(\sum_{n=1}^{N_m} q_{mnk} + \alpha_k - 1\right) \\ &= \frac{\theta_{mk}}{N_m + \sum(\alpha_{k'}-1)}\frac{\partial \log p}{\partial \theta_{mk}}\end{aligned}$$

Vectorize and can be re-written as:

$$\theta_m^{+} = \theta_m + \frac{1}{N_m + \sum(\alpha_{k'}-1)}\left[\text{diag}(\theta_m) - \theta_m\theta_m^T\right]\frac{\partial \log p}{\partial \theta_m} \tag{42}$$

For word per topic

$$\begin{aligned}\phi_{kv}^{+} &= \frac{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk}\delta(v-w_{mn}) + \beta_v - 1}{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk} + \sum(\beta_{v'}-1)} \\ &= \frac{\phi_{kv}}{\sum_{m=1}^{M}\sum_{n=1}^{N_m} q_{mnk} + \sum(\beta_{v'}-1)}\frac{\partial \log p}{\partial \phi_{kv}}\end{aligned}$$

Vectorize and can be re-written as:

$$\theta_m^{+} = \theta_m + \frac{1}{N_m + \sum(\alpha_{k'}-1)}\left[\text{diag}(\theta_m) - \theta_m\theta_m^T\right]\frac{\partial \log p}{\partial \theta_m} \tag{43}$$

C.4 SEM for LDA

We summarize our SEM derivation for LDA as follows:

E-Step

$$q_{mnk} = \frac{\theta_{mk}\phi_{kw_{mn}}}{\sum_{k'=1}^{K}\theta_{mk'}\phi_{k'w_{mn}}} \tag{44}$$

S-step

$$z_{mn} \sim \mathsf{Categorical}(q_{mn1}, ..., q_{mnK}) \tag{45}$$

M-step

$$\begin{aligned}\theta_{mk} &= \frac{D_{mk} + \alpha_k - 1}{N_m + \sum(\alpha_{k'}-1)} \\ \phi_{kv} &= \frac{W_{kv} + \beta_v - 1}{T_k + \sum(\beta_{v'}-1)}\end{aligned} \tag{46}$$

C.5 Equivalency

In case of LDA, let us consider only θ for the purpose of illustration. Now consider the case of stochastic EM, the update over one step is:

$$\theta_{ik}^{+} = \frac{n_{ik}}{N_i} = \frac{1}{N_i} \sum_{j=1}^{N_i} \delta(z_{ij} = k)$$

Again vectorizing and re-writing as earlier:

$$\theta_i^{+} = \theta_i + Mg$$

where $M = \frac{1}{N_i} \left[\text{diag}(\theta_i) - \theta_i \theta_i^T \right]$ and $g = \frac{1}{\theta_{ik}} \sum_{j=1}^{N_i} \delta(z_{ij} = k)$. The vector g can be shown to be an unbiased noisy estimate of the gradient, i.e.

$$\begin{aligned} \mathbb{E}[g] &= \frac{1}{\theta_{ik}} \sum_{j=1}^{N_i} \mathbb{E}[\delta(z_{ij} = k)] \\ &= \frac{1}{\theta_{ik}} \sum_{j=1}^{N_i} q_{ijk} \qquad\qquad = \frac{\partial \log p}{\partial \theta_{ik}} \end{aligned}$$

Thus, it is SGD with constraints. However, note that stochasticity does not arise from sub-sampling data as usually in SGD, rather from the randomness introduced in the S-step.

D Convergence

We now address the critical question of how the invariant measure of ESCA for the model presented in Section 2.1 is related to the true MAP estimates. First, note that SCA is ergodic [13], a result that immediately applies if we ignore the deterministic components of our automata (corresponding to the observations). Now that we have established ergodicity, we next study the properties of the stationary distribution and find that the modes correspond to MAP estimates.

We make a few mild assumptions about the model:

- The observed data Fisher information is non-singular, i.e. $I(\eta) \succ 0$.
- For the Fisher information for $\mathbf{z}|\mathbf{x}$, we need it to be non-singular and central limit theorem, law of large number to hold, i.e. $\mathbb{E}_{\eta_0}[I_Z(\eta_0)] \succ 0$ and
$$\sup_{\eta} \left| \frac{1}{n} \sum_{i=1}^{n} I_{z_i}(\eta) - \mathbb{E}_{\eta_0}[I_X(\eta)] \right| \to 0 \text{ as } n \to \infty$$
- We assume that $\frac{1}{n} \sum_{i=1}^{n} \nabla_\eta \log p(x_i; \eta) = 0$ has at least one solution, let $\hat{\eta}$ be a solution.

These assumptions are reasonable. For example in case of mixture models (or topic models), it just means all component must be exhibited at least once and all components are unique. The details of this case are worked out in Appendix E. Also when the number of parameters grow with the data, e.g., for topic models, the second assumption still holds. In this case, we resort to corresponding result from high dimensional statistics by replacing the law of large numbers with Donsker's theorem and everything else falls into place.

Consequently, we show ESCA converges weakly to a distribution with mean equal to some root of the score function ($\nabla_\eta \log p(x_i; \eta)$) and thus a MAP fixed point by borrowing the results known for SEM [16]. In particular, we have:

Theorem 1 *Let the assumptions stated above hold and $\tilde{\eta}$, is the estimate from ESCA. Then as the number of* i.i.d. *data point goes to infinity, i.e. $n \to \infty$, we have*

$$\sqrt{n}(\tilde{\eta} - \hat{\eta}) \xrightarrow{\mathfrak{D}} \mathcal{N}\left(0, I(\eta_0)^{-1}[I - F(\eta_0)^{-1}\right) \tag{47}$$

where $F(\eta_0) = I + \mathbb{E}_{\eta_0}[I_X(\eta_0)](I(\eta_0) + \mathbb{E}_{\eta_0}[I_X(\eta_0)])$.

This result implies that SEM flocks around a stationary point under very reasonable assumptions and tremendous computational benefits.

E Non-singularity of Fisher Information for Mixture Models

Let us consider a general mixture model:

$$p(x|\theta,\phi) = \sum_{k=1}^{K} \theta_k f(x|\phi_k) \tag{48}$$

Then the log-likelihood can be written as:

$$\log p(x|\theta,\phi) = \log\left(\sum_{k=1}^{K} \theta_k f(x|\phi_k)\right) \tag{49}$$

The Fisher Information is given by:

$$\begin{aligned} I(\theta,\phi) &= \mathbb{E}\left[(\nabla \log p(x|\theta,\phi))(\nabla \log p(x|\theta,\phi))^T\right] \\ &= \begin{bmatrix} \frac{\partial}{\partial\theta}\log p(x|\theta,\phi) \\ \frac{\partial}{\partial\phi}\log p(x|\theta,\phi) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial\theta}\log p(x|\theta,\phi) \\ \frac{\partial}{\partial\phi}\log p(x|\theta,\phi) \end{bmatrix}^T \end{aligned}$$

These derivatives can be computed as follows:

$$\begin{aligned} \frac{\partial}{\partial\theta_k}\log p(x|\theta,\phi) &= \frac{\partial}{\partial\theta_k}\log\left((\sum_{k=1}^{K} \theta_k f(x|\phi_k)\right) \\ &= \frac{f(x|\phi_k)}{\sum_{k'=1}^{K}\theta_{k'} f(x|\phi_{k'})} \\ \frac{\partial}{\partial\phi_k}\log p(x|\theta,\phi) &= \frac{\partial}{\partial\phi_k}\log\left((\sum_{k=1}^{K} \theta_k f(x|\phi_k)\right) \\ &= \frac{\theta_k \frac{\partial}{\partial\phi_k} f(x|\phi_k)}{\sum_{k'=1}^{K}\theta_{k'} f(x|\phi_{k'})} \end{aligned} \tag{50}$$

For any $u, v \in \mathbb{R}^K$ (with at least one nonzero), then the Fisher Information is positive definite as:

$$\begin{aligned} (u^T\ v^T) I \begin{pmatrix} u \\ v \end{pmatrix} &= (u^T\ v^T)\mathbb{E}\left[\begin{bmatrix} \frac{\partial}{\partial\theta}\log\left(\sum_{k=1}^{K}\theta_k f(X|\phi_k)\right) \\ \frac{\partial}{\partial\phi}\log\left(\sum_{k=1}^{K}\theta k f(X|\phi_k)\right) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial\theta}\log\left(\sum_{k=1}^{K}\theta_k f(X|\phi_k)\right) \\ \frac{\partial}{\partial\phi}\log\left(\sum_{i=1}^{K}\theta_k f(X|\phi_k)\right) \end{bmatrix}^T\right] \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \mathbb{E}\left[\left(u^T\frac{\partial}{\partial\theta}\log\left(\sum_{k=1}^{K}\theta_k f(X|\phi_k)\right) + v^T\frac{\partial}{\partial\theta}\log\left(\sum_{i=1}^{K}\theta_k f(X|\phi_i)\right)\right)^2\right] \\ &= \mathbb{E}\left[\left(\frac{\sum_{k=1}^{K} u_k f(X|\phi_k) + v_k\theta_k\frac{\partial}{\partial\phi_k}f(X|\phi_k)}{\sum_{k=1}^{K}\theta_k f(X|\phi_k)}\right)^2\right] \end{aligned}$$

This can be 0 if and only if

$$\sum_{k=1}^{K} u_k f(x|\phi_i) + v_k\theta_k\frac{\partial}{\partial\phi_k}f(x;\phi_k) = 0 \quad \forall x. \tag{51}$$

In case of exponential family emission models this cannot hold if all components are unique and all $\theta_k > 0$. Thus, if we assume all components are unique and every component has been observed at least once, the Fisher information matrix becomes non-singular.

F Alias Sampling Method

The alias sampling method is an efficient method for drawing samples from a K outcome discrete distribution in $O(1)$ amortized time and we describe it here for completeness. Denote by p_i for $i \in \{1 \dots K\}$ the probabilities of a distribution over K outcomes from which we would like to sample. If p were the uniform distribution, i.e. $p_i = K^{-1}$, then sampling would be trivial. For the general case, we must pre-process the distribution p into a table of K triples of the form (i, j, π_i) as follows:

- Partition the indices $\{1 \dots K\}$ into sets U and L where $p_i > K^{-1}$ for $i \in U$ and $p_i \leq K^{-1}$ for $i \in L$.
- Remove any i from L and j from U and add (i, j, p_i) to the table.
- Update $p_j = p_i + p_j - K^{-1}$ and if $p_j > K^{-1}$ then add j to U, else to L.

By construction the algorithm terminates after K steps; moreover, all probability mass is preserved either in the form of π_i associated with i or in the form of $K^{-1} - \pi_i$ associated with j. Hence, sampling from p can now be accomplished in constant time:

- Draw (i, j, π_i) uniformly from the set of k triples in K.
- With probability $K\pi_i$ emit i, else emit j.

Hence, if we need to draw from p at least K times, sampling can be accomplished in amortized $O(1)$ time.

G Applicability of ESCA

We begin with a simple Gaussian mixture model (GMM) with K components. Let $x_1, ..., x_n$ be *i.i.d.* observations, $z_1, ..., z_n$ be hidden component assignment variable and $\eta = \eta(\theta_1, ..., \theta_K, \mu_1, \Sigma_1, \mu_2, \Sigma_2, ..., \mu_K, \Sigma_K)$ be the parameters. Then the GMM fits into ESCA with sufficient statistics given by:

$$\begin{aligned} T(x_i, z_i) = [&\mathbb{1}\{z_i = 1\}, ..., \mathbb{1}\{z_i = K\}, \\ &x_i\mathbb{1}\{z_i = 1\}, ..., x_i\mathbb{1}\{z_i = K\}, \\ &x_i x_i^T\mathbb{1}\{z_i = 1\}, ..., x_i x_i^T\mathbb{1}\{z_i = K\}]. \end{aligned} \tag{52}$$

The conditional distribution for the E-step is:

$$p(z_i = k|x_i; \eta) \propto \theta_k \mathcal{N}(x_i|\mu_k, \Sigma_k) \tag{53}$$

In the S-step we draw from this conditional distribution and the M-step, through inversion of link function, is:

$$\begin{aligned} \tilde{\theta}_k &= \frac{1}{n + K\alpha - K}\sum_{i=1}^{n}(\mathbb{1}\{z_i = k\} + \alpha - 1) \\ \tilde{\mu}_k &= \frac{\kappa_0\mu_0 + \sum_{i=1}^{n} x_i\mathbb{1}\{z_i = k\}}{\kappa_0 + \sum_{i=1}^{n}\mathbb{1}\{z_i = k\}} \\ \tilde{\Sigma}_k &= \frac{\Psi_0 + \kappa_0\mu_0\mu_0^T + \sum_{i=1}^{n} x_i x_i^T\mathbb{1}\{z_i = k\} - (\kappa_0 + \sum_{i=1}^{n}\mathbb{1}\{z_i = k\})\tilde{\mu}_k\tilde{\mu}_k^T}{\nu_0 + d + 2 + \sum_{i=1}^{n}\mathbb{1}\{z_i = k\}} \end{aligned} \tag{54}$$

and is only function of the sufficient statistics.

Next, we provide more details on how to employ ESCA for any conditional exponential family mixture model; i.e., in which n random variables x_i, $i = 1, \ldots, n$ correspond to observations, each distributed according to a mixture of K components, with each component belonging to the same exponential family of distributions (e.g., all normal, all multinomial, etc.), but with different parameters:

$$p(x_i|\phi) = \exp(\langle\psi(x_i), \phi\rangle - g(\phi)). \tag{55}$$

The model also has n latent variables z_i that specify the identity of the mixture component of each observation x_i, each distributed according to a K-dimensional categorical distribution. A set of K mixture weights θ_k, $k = 1, \ldots, K$, each of which is a probability (a real number between 0 and 1 inclusive) and collectively sum to one. A Dirichlet prior on the mixture weights with hyper-parameters α. A set of K parameters ϕ_k, $k = 1, \ldots, K$, each specifying the parameter of the corresponding mixture component. For example, observations distributed according to a mixture of one-dimensional Gaussian distributions will have a mean and variance for each component. Observations distributed according to a mixture of V-dimensional categorical distributions (e.g., when each observation is a word from a vocabulary of size V) will have a vector of V probabilities, collectively summing to 1. Moreover, we put a shared conjugate prior on these parameters:

$$p(\phi; n_0, \psi_0) = \exp\left(\langle\psi_0, \phi\rangle - n_0 g(\phi) - h(m_0, \psi_0)\right). \tag{56}$$

Then joint sufficient statistics would be given by:

$$\begin{aligned} T(z_i, x_i) = [&\mathbb{1}\{z_i = 1\}, ..., \mathbb{1}\{z_i = K\}, \\ &\psi(x_i)\mathbb{1}\{z_i = 1\}, ..., \psi(x_i)\mathbb{1}\{z_i = K\}] \end{aligned} \tag{57}$$

In the E-step of t^{th} iteration, we derive the conditional distribution $p(z_i|x_i, \eta)$, namely

$$\begin{aligned} p(z_i = k|x_i, \eta) &\propto p(x_i|\phi_k^{t-1}, z_i = k)p(z_i = k|\theta^{t-1}) \\ &= \frac{\theta_k^{t-1}p(x_i|\phi_k^{t-1})}{\sum_{k'}\theta_{k'}^{t-1}p(x_i|\phi_{k'}^{t-1})} \end{aligned} \tag{58}$$

Table 3: Examples of some popular models to which ESCA is applicable.

mix. component/emitter	Bernoulli	Multinomial	Gaussian	Poisson
Categorical	Latent Class Model [6]	Unigram Document Clustering	Mixture of Gaussians [15]	
Dirichlet Mixture	Grade of Membership Model [22]	Latent Dirichlet Allocation [1]	Gaussian-LDA [4]	GaP Model [2]

In the S-step we draw z_i^t from this conditional distribution and the M-step through inversion of the link function yields:

$$\begin{aligned} \nabla g(\tilde{\phi}_k) &= \frac{\phi_0 + \sum_i \psi(x_i)\mathbb{1}\{z_i = k\})}{n_0 + \sum_i \mathbb{1}\{z_i = k\}} \\ \text{or} \qquad \tilde{\phi}_k &= \xi^{-1}\left(\frac{\psi_0 + \sum_i \psi(x_i)\mathbb{1}\{z_i = k\}}{n_0 + \sum_i \mathbb{1}\{z_i = k\}}\right) \\ \tilde{\theta}_k &= \frac{\sum_i \mathbb{1}\{z_i = k\} + \alpha_k - 1}{n + \sum_k \alpha_k - k} \,. \end{aligned} \tag{59}$$

This encompasses most of the popular mixture models (and with slight more work all the mixed membership or admixture models) with Binomial, multinomial, or Gaussian emission model, e.g. beta-binomials for identification, Dirichlet-multinomial for text or Gauss-Wishart for images as listed in Table 3.

Note further, ESCA is applicable to models such as restricted Boltzmann machines (RBMs) as well which are also in the exponential family. For example, if the data were a collection of images, each cell could independently compute the S-step for its respective image. For RBMs the cell would flip a biased coin for each latent variable, and for deep Boltzmann machines, the cells could perform Gibbs sampling.

To elabortate, consider 2-layer RBM (1 observed, 1 latent), then ESCA should work as it is. That is, we sample latent variables conditioned on data and weights. Then optimize weights, given latent variables and observed data. Now if we have deep RBM, i.e. one with many hidden layers. Then ESCA will have similar problem as Ising model. But there is a quick fix borrowing ideas from chromatic samplers.

for each iteration

1. Sample all odd layers of the RBM
2. Optimize for weights
3. Sample all even layers of the RBM
4. Optimize for weights

end for

We save a precise derivation and empirical evaluation for future work.

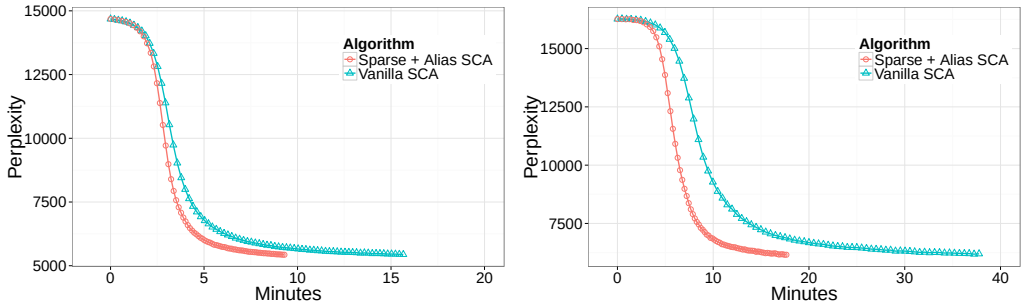
H More experimental results

In addition to the experiments reported in main paper, we perform another set of experiments. As before, to evaluate the strength and weaknesses of our algorithm, we compare against parallel and distributed implementations of CGS and CVB0.

Software & hardware All three algorithms were first implemented in the **Java** programming language. (We later switched to C++ for achieving better performance and those results are reported in the main paper.) To achieve good performance in the Java programming language, we use only arrays of primitive types and pre-allocate all of the necessary structures before the learning starts. We implement multithreaded parallelization within a node using the work-stealing Fork/Join framework, and the distribution across multiple nodes using the Java binding to OpenMPI. We also implemented a version of SCA with a sparse representation for the array D of counts of topics per documents and Vose's alias method to draw from discrete distributions. We run our experiments on a small cluster of 16 nodes connected through 10Gb/s Ethernet. Each node has two 8-core Intel Xeon E5 processors (some nodes have Ivy Bridge processors while others have Sandy Bridge processors) for a total of 32 hardware threads per node and 256GB of memory.

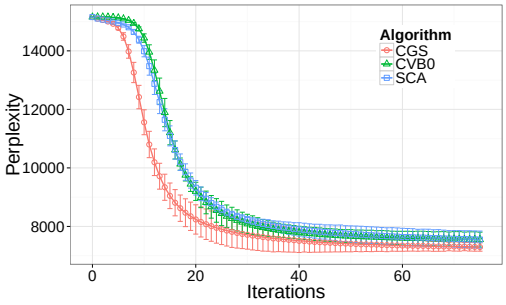
Datasets We experiment on two datasets, both of which are cleaned by removing stop words and rare words: Reuters RCV1 and English Wikipedia. Our Reuters dataset is composed of 806,791 documents comprising 105,989,213 tokens with a vocabulary of 43,962 vocabulary words. Our Wikipedia dataset is composed of 6,749,797 documents comprising 6,749,797 tokens with a vocabulary of 291,561 words. (Note this Wikipedia dump was collected at a different time than the main paper, hence different numbers.) We also apply the SCA algorithm to a third larger dataset composed of more than 3 billion documents comprising more than 171 billion tokens with a vocabulary of about 140,000 words.

Protocol We use perplexity on held-out documents to compare the algorithms. When comparing algorithms trained on Wikipedia, we compute the perplexity of 10,000 Reuters documents. Vice versa, when comparing algorithms trained on Reuters, we compute the perplexity of 10,000 Wikipedia documents. We run four sets of experiment on each dataset: (1) how perplexity evolves for some numbers of training iterations (100 topics); (2) how perplexity evolves over time (100 topics); (3) perplexity as a function of the number of topics (75 iterations); and (4) perplexity as a function of the value of β (100 topics, 75 iterations). With the exception of the second experiment, we ran all experiments five times with five different seeds, and report the mean and standard deviation of these runs. The results are presented in Figure 6. We also ran an experiment to compare vanilla SCA and its improved version that uses a sparse representation and Vose's alias method for discrete sampling. The results are presented in Figure 5.

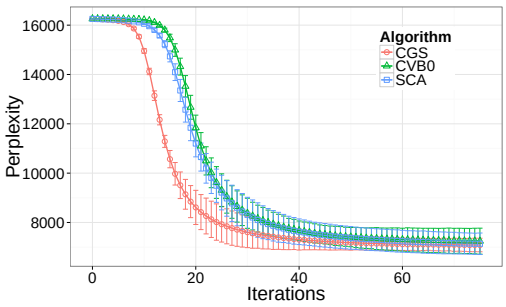


(a) Wikipedia, $K = 200$, $\alpha = 0.1$, $\beta = 0.1$ (b) Wikipedia, $K = 500$, $\alpha = 0.1$, $\beta = 0.1$

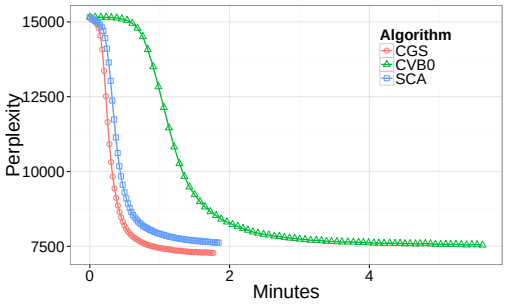
Figure 5: Evolution of perplexity over time for plain SCA and a sparse one using the alias method.



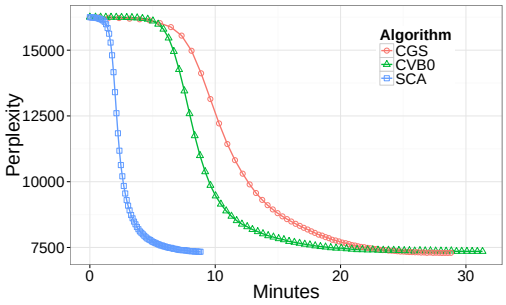
(a) Reuters, $K = 100$, $\alpha = 0.1$, $\beta = 0.1$



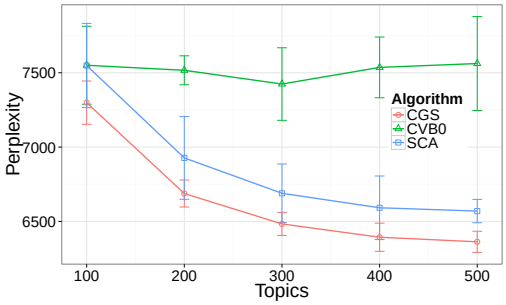
(b) Wikipedia, $K = 100$, $\alpha = 0.1$, $\beta = 0.1$



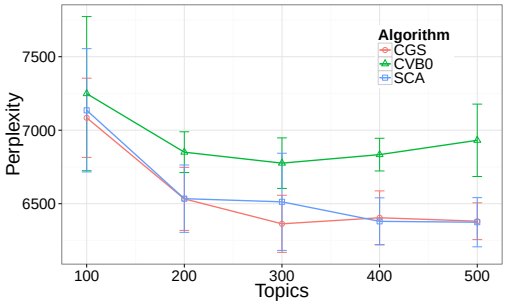
(c) Reuters, $K = 100$, $\alpha = 0.1$, $\beta = 0.1$



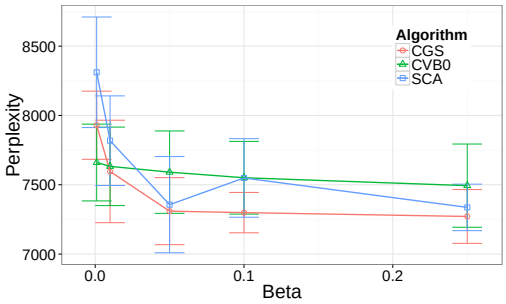
(d) Wikipedia, $K = 100$, $\alpha = 0.1$, $\beta = 0.1$



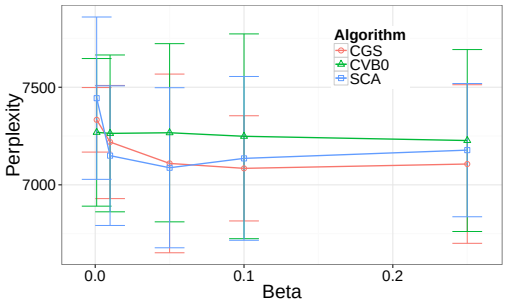
(e) Reuters, $\alpha = 0.1$, $\beta = 0.1$



(f) Wikipedia, $\alpha = 0.1$, $\beta = 0.1$



(g) Reuters, $K = 100$, $\alpha = 0.1$



(h) Wikipedia, $K = 100$, $\alpha = 0.1$

Figure 6: Evolution of perplexity on Wikipedia and Reuters over number of iterations, time, number of topics, value of β. Here SCA does not use alias method or sparsity and hence slower.

Topics

Here are the first five topics inferred via ESCA on LDA from both PubMed and Wikipedia:

PubMed				
Topic 0	Topic 1	Topic 2	Topic 3	Topic 4
seizures	data	local	gene	state
epilepsy	information	block	transcript	change
seizure	available	lidocaine	exon	transition
epileptic	provide	anethesia	genes	states
temporal_lobe	regarding	anethetic	expression	occur
anticonvulsant	sources	acupuncture	region	process
convulsion	literature	bupivacaine	mrna	shift
kindling	concerning	anaesthesia	mouse	condition
partial	limited	under	expressed	changed
generalized	provided	anaesthetic	human	dynamic
Wikipedia				
Topic 0	Topic 1	Topic 2	Topic 3	Topic 4
hockey	medical	von	boy	music
ice	medicine	german	youth	music
league	hospital	karl	boys	pop
played	physician	carl	camp	music
junior	doctor	friedrich	girl	artists
nhl	clinical	wilhelm	scout	electronic
professional	md	johann	girls	duo
games	physicians	ludwig	guide	genre
playing	doctors	prussian	scouts	genres
national	surgeon	heinrich	scouting	musicians